Almost rejectionless sampling from Nakagami-m distributions ($m \geq 1$)

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The Nakagami-m distribution is widely used for the simulation of fading channels in wireless communications. A novel, simple and extremely efficient acceptance-rejection algorithm is introduced for the generation of independent Nakagami-m random variables. The proposed method uses another Nakagami density with a half-integer value of the fading parameter, $m_p = n/2 \leq m$, as proposal function, from which samples can be drawn exactly and easily. This novel rejection technique is able to work with arbitrary values of $m \geq 1$, average path energy, Ω, and provides a higher acceptance rate than all currently available methods.

Introduction: The Nakagami-m distribution is widely used to model the wireless fading channel because of its good agreement with empirical channel measurements for some urban multipath environments [1]. The Nakagami probability density function (PDF) is $p_o(x) = C_p p(x)$, with $C_p = 2m^m/[\Omega\Gamma(m)]$ and

$$p(x) = x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right), \qquad x \geq 0 \tag{1}$$

where $m \geq 0.5$ is the fading parameter, which indicates the fading depth, and $\Omega > 0$ is the average received power.

Several schemes for drawing samples from a Nakagami-m PDF have been proposed. On the one hand, when m is an integer or half-integer (i.e. $m = n/2$ with $n \in \mathbb{N}$), independent samples can be generated through the square root of a sum of squares of n zero-mean independent identically distributed (IID) Gaussian random variables (RVs). On the other hand, for $m \neq n/2$ several techniques have been proposed for drawing correlated samples from (1) [2–4], but all of them present limitations in terms of complexity, applicability or poor performance for some values of m. Alternatively, several simple and efficient acceptance-rejection methods, using different proposals and with increasing accuracy, have been recently introduced [5–7]. Currently, the best results are provided by [7] using a truncated Gaussian PDF as the proposal.

In this Letter we provide an extremely efficient acceptance-rejection method for drawing independent samples from non-truncated (i.e. without any restriction in the domain) Nakagami PDFs with $m \geq 1$. As a proposal, we consider another Nakagami PDF with an integer or half-integer fading parameter, $m_p = n/2 \leq m$, from which samples can be easily and efficiently drawn [8]. Our approach is valid for arbitrary values of the fading parameters $m \geq 1$ (for many practical channels $1 \leq m \leq 15$, as discussed in [9]) and $\Omega > 0$. Furthermore, since our proposal is another Nakagami PDF, the novel rejection scheme provides a very good fit of the target, thus achieving very high acceptance rates that tend to 100% (i.e. exact or rejectionless sampling) when $m \to +\infty$ and outperforming all the alternative techniques reported in the literature.

Acceptance-rejection algorithm: Rejection sampling (RS) is a classical technique for generating samples from an arbitrary target PDF, $p_o(x) = C_p p(x)$ with $x \in \mathcal{D}$ and $C_p = [\int_{\mathcal{D}} p(x)dx]^{-1}$, using an alternative simpler proposal PDF, $\pi_o(x) = C_\pi \pi(x)$ with $x \in \mathcal{D}$ and $C_\pi = [\int_{\mathcal{D}} \pi(x)dx]^{-1}$, such that $\pi(x) \geq p(x)$, i.e. $\pi(x)$ is a hat function w.r.t. $p(x)$. RS works by generating samples from the proposal density, $x' \sim \pi_o(x)$, accepting them when $u' \leq p(x')/\pi(x')$, with u' uniformly distributed in [0,1], and rejecting them otherwise. The key performance measure for RS is the average acceptance rate, $a_R = \int_{\mathcal{D}} p(x)/\pi(x)\pi_o(x)dx = C_\pi/C_p \leq 1$. The value of a_R depends on how close the proposal is to the target, and determines the efficiency of the approach. Hence, the main difficulty when designing an RS algorithm is finding a good hat function, $\pi(x) \geq p(x)$, such that $\pi(x)$ and $p(x)$ are as close as possible and drawing samples from $\pi_o(x) = C_\pi \pi(x)$ can be done easily and efficiently.

In this work, we consider as target density the PDF given by (1) with $m \geq 1$. As proposal PDF, we suggest using another Nakagami function with different parameters, namely

$$\pi_o(x) \propto \pi(x) = \alpha_p x^{2m_p-1} \exp\left(-\frac{m_p}{\Omega_p}x^2\right), \qquad x \geq 0 \tag{2}$$

with $m_p = n/2$, $n = \lfloor 2m \rfloor$ (with $\lfloor x \rfloor$ denoting the integer part of $x \in \mathbb{R}$), and the remaining parameters (α_p and Ω_p) adjusted to obtain the same location and value of the maximum in the proposal as in the target:

$$\Omega_p = \frac{2m_p}{2m_p - 1}x_{\max}^2 = \Omega\frac{m_p(2m-1)}{m(2m_p-1)} \tag{3}$$

$$\begin{aligned}\alpha_p &= \frac{p(x_{\max})}{x_{\max}^{2m_p-1}\exp\left(-m_p x_{\max}^2/\Omega_p\right)} \\ &= \exp(m_p - m)\left(\frac{\Omega(2m-1)}{2m}\right)^{m-m_p}\end{aligned} \tag{4}$$

where $x_{\max}$ is the location of the maximum of the Nakagami PDF, obtained solving $dp(x)/dx = 0$, which results in

$$x_{\max} = \sqrt{\frac{(2m-1)\Omega}{2m}} \tag{5}$$

Note that we always have $m_p \leq m$, with m_p being an integer or half-integer value. Thanks to this choice of m_p and the parameters derived in (3) and (4), we can ensure that: (a) we can draw samples exactly from $\pi_o(x) \propto \pi(x)$ [8]; (b) $\pi(x) \geq p(x)$ for all $x \geq 0$, as proved in the sequel. Fig. 1*a* shows an example of the target, $p(x)$, our proposal, $\pi(x)$, and the proposal used in [7] for an unbounded domain, which fits the true PDF in a much looser way than ours, thus leading to worse acceptance rates.

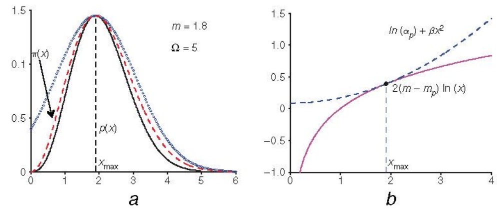

Fig. 1 *Example (Fig. 1a) of Nakagami PDF (solid line) with $m = 1.8$ and $\Omega = 5$, our proposal (dashed line), and Gaussian proposal, $\pi(x) = p(x_{max}) \exp(-m/\Omega(x - x_{max})^2)$ used in [7] for an unbounded domain (circles). The two functions (Fig. 1b), $\ln(\alpha_p) + \beta x^2$ (dashed line) and $2(m - m_p)\ln(x)$ (solid line) in (9), when $m = 1.8$ and $\Omega = 5$*

Therefore, our algorithm follows these three simple steps: (a) calculate the parameters of the proposal PDF, $\pi_o(x) \propto \pi(x)$; (b) draw a sample x' from $\pi_o(x)$ using the direct approach described in [8]: generate $2m_p$ IID Gaussian RVs, $z_k \sim \mathcal{N}(0, 1)$ for $1 \leq k \leq 2m_p$, and set

$$x' = \sqrt{\frac{\Omega_p}{2m_p}\sum_{k=1}^{2m_p} z_k^2} \tag{6}$$

(c) accept x' with probability $p(x')/\pi(x')$ and discard otherwise. Steps (b) and (c) are repeated until the desired number of samples has been obtained.

Proof of RS inequality: To apply the RS technique we need to ensure that $\pi(x) \geq p(x)$, i.e.

$$\alpha_p x^{2m_p-1}\exp\left(-\frac{m_p x^2}{\Omega_p}\right) \geq x^{2m-1}\exp\left(-\frac{mx^2}{\Omega}\right) \quad \forall x \geq 0 \tag{7}$$

Alternatively, (7) can be easily rewritten as

$$\alpha_p \exp(\beta x^2) \geq x^{2(m-m_p)}, \qquad \forall x \geq 0 \tag{8}$$

where $\beta \triangleq m/\Omega - m_p/\Omega_p$ and $x^{2(m-m_p)}$ presents a sub-linear growth, since $0 \leq 2(m - m_p) < 1$. Finally, taking the logarithm on both sides of (8),

$$\ln\alpha_p + \beta x^2 \geq 2(m - m_p)\ln x, \qquad \forall x \geq 0 \tag{9}$$

Now, since $m \geq m_p$ and Ω_p is given by (3), we note that

$$\beta = \frac{m}{\Omega} - \frac{m_p}{\Omega_p} = \frac{m}{\Omega}\left(1 - \frac{2m_p - 1}{2m - 1}\right) \geq 0 \tag{10}$$

Hence, since we have $\alpha_p > 0$ from (4), the parabola on the left-hand side of (9) is an increasing function with an increasing first derivative (i.e. a convex function). Moreover, since $m \geq m_p$, the logarithmic function on the right-hand side of (9) is also an increasing function, but with a decreasing first derivative (i.e. a concave function). Consequently, since both functions are increasing for $x \geq 0$, but $\ln \alpha_p + \beta x^2$ is convex and $2(m - m_p) \ln x$ is concave, they can have at most two intersection points. However, as shown in Fig. 1*b*, the two functions are tangent at $x = x_{\max}$, which is the only contact point between both curves. To prove this, we need to show that both functions are equal at $x = x_{\max}$, i.e.

$$\begin{aligned} \ln \alpha_p + \beta x_{\max}^2 &= 2(m - m_p) \ln x_{\max} \\ &= (m - m_p) \ln\left(\frac{\Omega(2m-1)}{2m}\right) \end{aligned} \tag{11}$$

and also that their first derivatives are equal, i.e.

$$\begin{aligned} \left.\frac{d(\ln \alpha_p + \beta x^2)}{dx}\right|_{x=x_{\max}} &= \left.\frac{d(2(m - m_p) \ln x)}{dx}\right|_{x=x_{\max}} \\ &= \frac{2(m-m_p)}{x_{\max}} = \sqrt{\frac{8m(m-m_p)^2}{(2m-1)\Omega}} \end{aligned} \tag{12}$$

Therefore, since x^2 grows faster than $\ln x$, we can guarantee that $\ln \alpha_p + \beta x^2 \geq 2(m - m_p) \ln x \ \forall x \geq 0$, with equality only at $x = x_{\max}$, and the RS inequality in (7) is satisfied.

Results: To analyse the performance of the algorithm, we have compared the acceptance rate (AR), a_R, of our approach and the Gaussian proposal used in [7] to draw samples from a Nakagami PDF without truncation. The AR of our technique can be obtained analytically:

$$a_{R1} = (2e)^{m-m_p} \frac{\Gamma(m)(2m_p - 1)^{m_p}}{\Gamma(m_p)(2m-1)^m} \tag{13}$$

with $\Gamma(m)$ denoting the gamma function, whereas the AR for the proposal used in [7] can be approximated for $m \geq 4$ as

$$a_{R2} \simeq \frac{e^{m-1/2}\Gamma(m)(2m-1)^{1/2-m}}{\sqrt{\pi}2^{m+1/2}} \tag{14}$$

Note that in both cases the AR is independent of the average received power, Ω. Fig. 2 shows this AR, obtained empirically after drawing $N = 6 \times 10^5$ independent samples, for both approaches and several values of the fading depth, *m*. It can be seen that our technique is extremely efficient, outperforming the proposal used in [7] and providing the best results ever reported in the literature for $m \geq 1$. Furthermore, our technique provides exact sampling (i.e. $a_{R1} = 1$) when *m* is an integer or half-integer, since our proposal is equal to the target in these cases.

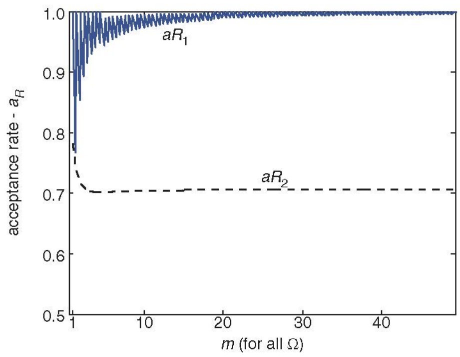

Fig. 2 *Acceptance rate (AR) using our proposal (continuous line) and the one from [7] for an unbounded domain (dashed line) for $1 \leq m \leq 50$*

Conclusion: We have proposed a rejection sampling (RS) scheme for generating Nakagami random variables, with arbitrary values of $m \geq 1$ and Ω, where the proposal PDF is itself another Nakagami-*m* density. The proposed algorithm is simple and extremely efficient, providing the best acceptance rates ever reported in the literature for $m \geq 1$.

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